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AN ANALYTICAL STUDY OF ORBITAL RENDEZVOUS FOR

LEAST FUEL AND LEAST ENERGY

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AN ANALYTICAL STUDY OF ORBITAL RENDEZVOUS FOR LEAST FUEL AND LEAST ENERGY[1]

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SUMMARY

The equations of relative motion with impulsive velocity changes are solved for rendezvous with a target in a circular orbit and for motion near the apsis of any conic target orbit. The maneuver times associated with least-fuel interception, least-fuel rendezvous, and least-energy rendezvous are determined from the solutions. The velocity impulse requirement for least-fuel interception is relatively distinct; however, that for least-fuel rendezvous is not. For this reason, often the time required for a rendezvous maneuver may be shortened significantly from that for least fuel without incurring a significant fuel penalty. It is shown that an economical rendezvous depends not only upon the time that is taken for the rendezvous maneuver from the position at which it is initiated; but in addition, it depends on the selection of the position in the uncorrected relative motion. From these considerations a guidance logic for economical rendezvous is evolved. This system is a multi-impulse scheme depending on a priori knowledge of the target orbit parameters and the measurement of the relative range and relative velocity vectors following target acquisition. The guidance logic permits an optimum position of initiation of the rendezvous maneuver to be determined. It is shown that the penalty for commencing a rendezvous maneuver at a position significantly different from the optimum position may be substantial.

INTRODUCTION

During the past several years, a number of proposals and studies have been made of space-directed missions, manned and unmanned. Many

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[1]Preliminary results of this study were presented by the author in a paper entitled "Least Fuel, Least Energy, and Salvo Rendezvous" which was delivered before the 15th Annual Spring Technical Conference of the IRE and ARS, Cincinnati, Ohio, April 13, 1961. The proceedings of this conference were not published by the sponsoring societies and the present report is being released to make the information on this study more generally available.

of these missions have involved, either implicity or explicitly, the rendezvousing of a maneuverable space vehicle with another orbiting object. However, a literature search reveals few attempts to study systematically the problems of the economical rendezvousing of space vehicles, although the specific problems of optimum interception (ref. 1), target orbital placement (ref. 2), and interceptor orbital placement (ref. 3) have been studied.

Rendezvous, of course, is an end in itself. But there are two practical examples of particular interest to space flight, namely, orbital refueling and orbital parking, in which the advantages of rendezvousing depend upon how cheaply it can be done. Cost is a problem since rendezvousing can be prohibitively expensive if we try to do things the wrong way. Suppose, for example, we try to correct an off-course error in too short a time. Then the fuel requirement can become an order of magnitude greater than the least-fuel solution to the problem may have allowed. A similar situation can arise if we try to correct an off-course error at the wrong position in the relative motion; the local least-fuel requirement may be an order of magnitude greater than at the best position.

This paper presents a study of the problem of economical rendezvous of a maneuvering space vehicle with an orbiting target vehicle. The principal analysis is based upon a maneuver that is idealized as a set of impulsive changes in the velocity of the interceptor, with the target either in a circular orbit or near an apsis of any conic orbit. The objectives are to obtain a guidance logic for economical rendezvousing and to obtain an understanding of the basic analytical features and the characteristics of a rendezvous maneuver.

SYMBOLS

D apsidal error term, $D^2 = \left(\frac{S^2}{S^2 - y^2}\right) F^2$

e target orbit eccentricity

$\underline{E}$ error velocity vector in reduced solution, equal to minimum-fuel interception velocity

F	error velocity in apsidal solution, equal to minimum-fuel interception velocity
h	target orbit altitude
l, m, n	direction cosines of the initial rendezvous impulse
l_o, m_o, n_o	direction cosines of the terminal rendezvous impulse at the interception point
m	mass of the interceptor
$o(x)$	less than the quantity x by at least an order of magnitude
$O(x)$	of the order of magnitude of the quantity x
O_x, O_y, O_z	orthogonal triad set up with its origin in the target (The outward radius vector through the target, $\underline{r}_o$, is the direction of the y axis, the z axis is perpendicular to the plane of the target's motion and has a positive sense in the direction of $\underline{\Omega}$.)
p	target orbit semilatus rectum
q	generalized coordinate equal to either x or y
$\underline{r}$	radius vector through the interceptor from the center of the force field
$\underline{r}_b$	radius vector of the interceptor at cutoff of booster
$\underline{r}_o$	radius vector through the target from the center of the force field
$\underline{S}_R$	relative range or line of sight vector from the target to the interceptor
$\dot{S}$	component of the relative velocity vector in the direction of the line of sight, $V \sin \gamma$
t	time
t_o	time to the apsis in the target orbit
$\underline{T}$	thrust vector of the interceptor

V_b	velocity of the interceptor at burnout
$\underline{V}_R$	relative velocity vector of the interceptor with respect to the target
x, y, z	components of $\underline{S}_R$ in the orthogonal triad
$\dot{x}, \dot{y}, \dot{z}$	components of $\underline{V}_R$ in the orthogonal triad
α	angle between the thrust vector and the initial line of sight vector
γ	angle between relative velocity vector, $\underline{V}_R$, and the perpendicular to the line of sight, $\underline{S}_R$
ΔV	sum of the absolute values of the initial and terminal impulses
$(\Delta V)_e$	sum of the absolute values of the initial and terminal impulses for a maneuver time corresponding to the least equivalent energy
$\frac{1}{2}(\Delta V_e)^2$	total equivalent energy of the rendezvous transfer impulses
$(\Delta V)_I$	sum of the absolute values of the initial and terminal impulses for a maneuver time corresponding to the least-fuel interception
ΔV_o	terminal impulse that must be made to bring the interceptor and target vehicles to relative rest at the interception point
ΔV_1	initial impulse in velocity of the interceptor that must be made in order to intercept the target
$\underline{\Omega}$	angular velocity vector of the target orbiting in the central force field
w	scalar magnitude of $\underline{\Omega}$
ρ	magnitude of the relative range in the coplanar case, $\rho^2 = x^2 + y^2$
σ	standard deviation of a given quantity, assuming that errors are distributed in a Gaussian fashion
τ	time taken for a rendezvous maneuver

τ_o time of the Hohman transfer from r_b to r_o

v target orbit true anomaly

μ mass gravitational constant of the central force field, GM

$(\dot{\ })$ rate of change

Subscripts

b cutoff of booster

e equivalent energy

I interception

o target, or terminal value at the target

R relative

1 initial

$(\underline{\ })$ vector quantity

ANALYSIS

Equations of Relative Motion

The rendezvous problem is defined by the boundary constraints that the position and velocity of a maneuvering space vehicle and an orbiting target vehicle are to be matched. It is assumed that the interceptor vehicle has been launched in the general direction of the target but is not absolutely on course (see appendix A), so that a corrective maneuver is necessary to insure an interception as indicated schematically in figure 1(a). Further, it is assumed the interceptor carries sensors that will acquire the target at some position along the initial launch path and measure the quantities needed to evaluate and determine subsequent maneuvers. Each maneuver is idealized as a set of impulsive changes in the velocity of the interceptor.

Let us consider the situation following target acquisition. We shall use the notation that the angular velocity vector of the orbiting target is $\underline{\Omega}$. The target "sees" the relative range or line of sight vector, $\underline{S}_R$, and the relative velocity vector, $\underline{V}_R$, through the interceptor.

The radius vector of the interceptor relative to the central force is $\underline{r}$. An orthogonal triad, Ox, Oy, Oz, has its origin in the target, as shown in figure 1(b). The outward radius vector through the target, $\underline{r}_o$, is the direction of the y axis. The z axis is perpendicular to the plane of the target's motion and has a positive sense in the direction of $\underline{\Omega}$. In the triad, $\underline{S}_R$ has the components x, y, z, $\underline{V}_R$ has the components $\dot{x}$, $\dot{y}$, $\dot{z}$, and $\underline{\Omega}$ has the components 0, 0, w. Since the target is in a planar force field, $\dot{\underline{\Omega}} = (\dot{w}/w)\underline{\Omega}$, with the components 0, 0, $\dot{w}$. Oblateness effects are neglected throughout this paper since they are insignificant to the relative motion but cause small changes in the magnitudes, w and $\dot{w}$, of the absolute motion vectors, $\underline{\Omega}$ and $\dot{\underline{\Omega}}$. In a real situation, the corrected absolute values of these quantities must be used (see, e.g., ref. 4).

The vector equation of relative motion is now:

$$\frac{d^2(\underline{S}_R + \underline{r}_o)}{dt^2} + 2\left[\underline{\Omega} \times \left(\frac{d\underline{r}_o}{dt} + \underline{V}_R\right)\right] + (\dot{\underline{\Omega}} \times \underline{r}) + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = \nabla\left(\frac{\mu}{r}\right) + \frac{\underline{T}}{m}$$

The thrust vector, $\underline{T}$, is included in the foregoing equation to show how it enters the equations of motion. One simple result will be obtained, however, before we set $\underline{T} = 0$ and consider only impulsive motion. If we assume that the target orbit is circular ($\dot{\underline{\Omega}} = 0$) and take the scalar product of the vector equation of motion with the relative velocity vector, $\underline{V}_R$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(V_R{}^2 + w^2z^2 - w^2r^2 - \frac{2\mu}{r}\right) = \frac{\underline{T}}{m}\cdot\frac{d\underline{S}_R}{dt}$$

For the special case of constant thrust, more precisely, when $\underline{T}/m$ is a constant vector, an integral of energy is available in the form:

$$\frac{\underline{T}}{m} \cdot \underline{S}_R = \frac{1}{2}\left(V_R{}^2 + w^2z^2 - w^2r^2 - \frac{2\mu}{r} + w^2r_o{}^2 + \frac{2\mu}{r_o}\right) \tag{1}$$

where the rendezvous end conditions (i.e., $\underline{V}_R = 0$ and $\underline{S}_R = 0$) have been imposed. Since,

$$r^2 = r_o{}^2 + 2yr_o + x^2 + y^2 + z^2 = (r_o + y)^2 + O(x^2)$$

an approximation that is valid when $S_R << r_o$ can be obtained

$$\frac{\underline{\underline{T}}}{m} \cdot \underline{S}_R = \left(\frac{T}{m}\cos\alpha\right)(S_R) \simeq \frac{1}{2}\left[V_R^2 + w^2\,(z^2 - 3y^2)\right] \tag{2}$$

where α is the angle between the thrust vector and the initial line of sight vector, $\underline{S}_R$.

To illustrate typical results with this equation, suppose we consider a situation where $S_R \sim 40$ miles, $V_R \sim 1000$ fps, and $w \sim 10^{-3}$/sec. We find that $(T/m)\cos\alpha \sim (1/10)g$. In general, the angle α will be small so that $(T/m) \sim (1/10)g$. For rendezvous with constant thrust, then, it is indicated that the thrust required is small; for example, if the interceptor weighs 5000 lb, the required thrust level is, for this case, only 500 lb. Rendezvous with impulsive thrust involves thrust levels two orders of magnitude greater than this continuous thrust solution (thrust accelerations of the order of 6 to 10 g delivered for periods of the order of 1 to 3 seconds typically). Hence, the character of any solutions that can be obtained from the continuous thrust problem may be expected to differ substantially from the impulsive thrust solutions that we will now proceed to develop.

To obtain the impulsive thrust solutions, we will set $\underline{T} = 0$. The scalar equations of motion are now:

$$\left.\begin{aligned} \ddot{x} - 2w\dot{y} - \dot{w}(y + r_o) - w^2 x &= -\frac{\mu}{r^3}x + 2w\dot{r}_o \\ \ddot{y} + 2w\dot{x} + \dot{w}x - w^2(y + r_o) &= -\frac{\mu}{r^3}(y + r_o) - \ddot{r}_o \\ \ddot{z} &= -\frac{\mu}{r^3}z \end{aligned}\right\} \tag{3}$$

where the coefficients in these equations may be identified to the target orbit parameters: e, the eccentricity; v, the true anomaly; p, the semi-latus rectum; and $w_o = \sqrt{\mu/p^3}$, through the equations,

$$\left.\begin{aligned} w &= w_o(1 + e\cos v)^2 \\ \dot{w} &= -2w_o^2\, e \sin v\,(1 + e\cos v)^3 \\ r_o &= p(1 + e\cos v)^{-1} \\ \frac{\mu}{r^3} &= w_o^2(1 + e\cos v)^3\left(1 + 2\frac{y}{r_o} + \frac{x^2 + y^2 + z^2}{r_o^2}\right)^{-3/2} \end{aligned}\right\} \tag{4}$$

Circular Target Orbits

Reduced equations of motion.- A variety of restrictions may now be imposed to yield a tractable problem. First, if the relative range is small compared to the orbit radius (i.e., $S_R << r_o$), then terms of order $(y/r_o)^2$ may be neglected and $r \approx r_o + y$. This assumption leads to the general first-order problem of relative motion. It does not lend itself to simple integration except in the case of a circular target orbit. The solution to this case is derived in appendix B and appears to be too cumbersome for further direct analysis. However, an inspection of the table in appendix A leads to the conclusion that practical problems can occur in which the difference in gravity between target and interceptor, within the miss-distance sphere of uncertainty, is less than 1 percent. Thus, as an additional assumption, we neglect the change in gravity over the altitude range of interest such that $w_o y$ is, in effect, small compared to $\dot{x}$. With these approximations and if the target orbit is assumed circular, the equations become simply,

$$\left.\begin{aligned} \ddot{x} - 2w_o\dot{y} &= 0 \\ \ddot{y} + 2w_o\dot{x} &= 0 \\ \ddot{z} + w_o^2 z &= 0 \end{aligned}\right\} \tag{5}$$

and integrate immediately to:

$$\left.\begin{aligned} x &= \left(x_o + \frac{\dot{y}_o}{2w_o}\right) + \frac{\dot{x}_o}{2w_o}\sin 2w_o t - \frac{\dot{y}_o}{2w_o}\cos 2w_o t \\ y &= \left(y_o - \frac{\dot{x}_o}{2w_o}\right) + \frac{\dot{y}_o}{2w_o}\sin 2w_o t + \frac{\dot{x}_o}{2w_o}\cos 2w_o t \\ z &= \frac{\dot{z}_o}{w_o}\sin w_o t + z_o \cos w_o t \end{aligned}\right\} \tag{6}$$

where the boundary conditions, $\underline{S_R} = (x_o, y_o, z_o)$, $\underline{V_R} = (\dot{x}_o, \dot{y}_o, \dot{z}_o)$, have been applied at $t = 0$.

Velocity requirements.- From equations (6), the velocity requirements of the rendezvous maneuver can be determined. In particular, if we set $x_o = y_o = z_o = 0$ and compute in negative time, the vector $\underline{V_{R_o}} = (\dot{x}_o, \dot{y}_o, \dot{z}_o)$ represents the terminal impulse produced by rendezvousing in time $-t$ from the initial position $\underline{S_R} = (x, y, z)$.

The initial impulse is the difference between the initial velocity $\underline{V}_{R_1} = (\dot{x}_1, \dot{y}_1, \dot{z}_1)$ at $\underline{S}_R = (x, y, z)$, and the velocity needed to achieve an interception in time $-t$, $\underline{V}_R = (\dot{x}, \dot{y}, \dot{z})$. The final impulse then has these components:

$$\left.\begin{aligned}
\frac{\dot{x}_o}{w_o} &= \frac{x \cos w_o t - y \sin w_o t}{\sin w_o t} = -x \cot w_o\tau - y \\
\frac{\dot{y}_o}{w_o} &= \frac{x \sin w_o t + y \cos w_o t}{\sin w_o t} = x - y \cot w_o\tau \\
\frac{\dot{z}_o}{w_o} &= \frac{z}{\sin w_o t} = -\frac{z}{\sin w_o\tau}
\end{aligned}\right\} \tag{7a}$$

while the velocity required at $-t$ has these components:

$$\left.\begin{aligned}
\frac{\dot{x}}{w_o} &= \frac{x \cos w_o t + y \sin w_o t}{\sin w_o t} = -x \cot w_o\tau + y \\
\frac{\dot{y}}{w_o} &= \frac{-x \sin w_o t + y \cos w_o t}{\sin w_o t} = -x - y \cot w_o\tau \\
\frac{\dot{z}}{w_o} &= z\,\frac{\cos w_o t}{\sin w_o t} = -z \cot w_o\tau
\end{aligned}\right\} \tag{7b}$$

where $\tau = -t$.

Equations (7a) and (7b) may be expressed vectorially:

$$\underline{V}_{R_o} = \left(\underline{\Omega} \times \underline{S}_R\right) - \left(w_o \cot w_o\tau\right)\underline{S}_R - \left(z \tan \frac{w_o\tau}{2}\right)\underline{\Omega} \tag{7c}$$

$$\underline{V}_R = -\left(\underline{\Omega} \times \underline{S}_R\right) - \left(w_o \cot w_o\tau\right)\underline{S}_R \tag{7d}$$

The condition for a collision in time τ is equation (7d). For the target to be intercepted in time τ, therefore, a vector impulse change $\underline{\Delta V}_1$ to the initial relative velocity vector $\underline{V}_{R_1}$ must be made, given by,

$$\underline{\Delta V}_1 = \underline{V}_R - \underline{V}_{R_1}$$

$$= -(\underline{\Omega} \times \underline{S}_R) - (w_o \cot w_o\tau)\underline{S}_R - \left(\frac{\underline{V}_{R_1} \cdot \underline{S}_R}{S^2}\right)\underline{S}_R - \left(\frac{\underline{S}_R \times \underline{V}_{R_1}}{S}\right) \times \frac{\underline{S}_R}{S}$$

$$= -\left(w_o \cot w_o\tau + \frac{\dot{S}}{S}\right)\underline{S}_R - (\underline{\Omega} \times \underline{S}_R) - \left(\frac{\underline{S}_R \times \underline{V}_{R_1}}{S}\right) \times \frac{\underline{S}_R}{S}$$

where S is the magnitude of the relative range vector, $\underline{S}_R$, and $\dot{S}$ is the component of the relative velocity along $\underline{S}_R$. This component is given by $\dot{S} = V \sin\gamma$ where $\pi/2 + \gamma$ is the angle between the relative velocity vector, $\underline{V}_{R_1}$ (magnitude V), and the line of sight, $\underline{S}_R$. Let us define a vector $\underline{E}$ by

$$\underline{E} = (\underline{\Omega} \times \underline{S}_R) + \left(\frac{\underline{S}_R \times \underline{V}_{R_1}}{S}\right) \times \frac{\underline{S}_R}{S} \tag{8}$$

then $\underline{E}$ is perpendicular to $\underline{S}_R(\underline{E} \cdot \underline{S}_R = 0)$, does not contain τ, and has a scalar magnitude given by,

$$E^2 = (\underline{\Omega} \times \underline{S}_R)^2 + 2\left(\underline{\Omega}\ \underline{S}_R\ \underline{V}_{R_1}\right) + \left(\frac{\underline{S}_R \times \underline{V}_{R_1}}{S}\right)^2$$

$$= w_o{}^2(S^2 - z^2) - 2w_o(\dot{x}_1 y - x\dot{y}_1) + V^2 - \dot{S}^2$$

Thus, we have a decomposition of $\underline{\Delta V}_1$ along the two perpendicular directions, $\underline{S}_R$ and $\underline{E}$:

$$\underline{\Delta V}_1 = (w_o S \cot w_o\tau + \dot{S})\frac{\underline{S}_R}{S} - \underline{E} \tag{9a}$$

with

$$(\underline{\Delta V}_1)^2 = (w_o S \cot w_o\tau + \dot{S})^2 + E^2 \tag{9b}$$

If the two vehicles are already on a collision course,

$$\underline{\Delta V}_1 = 0$$

and the two terms in the orthogonal decomposition must be independently zero. The first term being zero yields the results:

$$\cot w_o\tau = -\frac{\dot{S}}{w_oS} = -\frac{\dot{x}_1 - w_oy}{w_ox} = -\frac{\dot{y}_1 + w_ox}{w_oy} = -\frac{\dot{z}_1}{w_oz}$$

The second term being zero yields, obviously, $\underline{E} = 0$. This vector $\underline{E}$ is, in essence, an error velocity. As noted, it is zero if the vehicles are on a collision course. In addition, the velocity $\underline{E}$ represents the least-fuel impulse to effect an interception of the target. This result can be demonstrated with equations (9) if it is noted that the time to intercept τ is a free variable which may be used to make $w_oS \cot w_o\tau + \dot{S} = 0$.

The magnitude of the terminal impulse, $\underline{\Delta V}_o = -\underline{V}_{R_o}$, that must be made to bring the vehicles to relative rest at the interception point may be computed from the components of $\underline{V}_{R_o}$ given by equations (7a):

$$(\Delta V_o)^2 = \dot{x}_o^{\,2} + \dot{y}_o^{\,2} + \dot{z}_o^{\,2} = \frac{w_o^2(x^2 + y^2 + z^2)}{\sin^2 w_o\tau}$$

or

$$\Delta V_o = \frac{w_oS}{\sin w_o\tau} \tag{10}$$

Minimum fuel and minimum energy. - To minimize the fuel expended in the rendezvous maneuver, it is necessary to minimize the sum of absolute values of the initial and terminal impulses. We form,

$$\Delta V = \Delta V_o + \Delta V_1 = \frac{w_oS}{\sin w_o\tau} + \sqrt{(w_oS \cot w_o\tau + \dot{S})^2 + E^2}$$

and note that ΔV is a function of $w_o\tau$, the angular rotation of the target during the maneuver, which we will call the true rendezvous anomaly. Let us differentiate with respect to $w_o\tau$:

$$\frac{\partial \Delta V}{\partial(w_o\tau)} = -\frac{w_o S \cos w_o\tau}{\sin^2 w_o\tau} - \frac{(w_o S \cot w_o\tau + \dot{S}) w_o S}{\sin^2 w_o\tau \sqrt{(w_o S \cot w_o\tau + \dot{S})^2 + E^2}}$$

Stationary values occur whenever

$$\frac{\cot w_o\tau}{\sqrt{1 + \cot^2 w_o\tau}} = -\frac{\cot w_o\tau + \dfrac{\dot{S}}{w_o S}}{\sqrt{\left(\cot w_o\tau + \dfrac{\dot{S}}{w_o S}\right)^2 + \dfrac{E^2}{w_o^2 S^2}}}$$

that is,

$$\cot w_o\tau = -\frac{\dot{S}}{w_o S + E} \tag{11}$$

If the true rendezvous anomaly, $w_o\tau$, is determined from the stationary value given by equation (11), then the rendezvous maneuver will involve the least amount of fuel since

$$\frac{\partial^2 \Delta V}{\partial(w_o\tau)^2} = \left[\frac{w_o^4 S^4}{(\Delta V_o)^3} + \frac{w_o^2 S^2 E^2}{(\Delta V_1)^3}\right]\left[\frac{d \cot (w_o\tau)}{d(w_o\tau)}\right]^2 > 0$$

at this value of $w_o\tau$.

In a similar manner, let us consider next the total equivalent energy of the transfer and form,

$$(\Delta V_e)^2 = (\Delta V_o)^2 + (\Delta V_1)^2$$

$$= 2w_o^2 S^2 \cot^2 w_o\tau + 2w_o S\dot{S} \cot w\tau + E^2 + \dot{S}^2 + w_o^2 S^2$$

Differentiation with respect to $w_o\tau$,

$$\frac{\partial(\Delta V_e)^2}{\partial(w_o\tau)} = \left(4w_o^2 S^2 \cot w_o\tau + 2w_o S\dot{S}\right) \frac{d \cot (w_o\tau)}{d(w_o\tau)}$$

shows that stationary values occur whenever

$$\cot w_o\tau = -\frac{\dot{S}}{2w_oS} \tag{12}$$

and at these stationary values,

$$\frac{\partial^2(\Delta V_e)^2}{\partial(w_o\tau)^2} = 4w_o^2 S^2\left[\frac{d \cot (w_o\tau)}{d(w_o\tau)}\right]^2 > 0$$

Thus, equation (12) is the condition for least total energy in rendezvousing with the target.

Next, consider the initial impulse ΔV_1 only; from equations (9), it is a minimum when

$$\cot w_o\tau = -\frac{\dot{S}}{w_oS} \tag{13}$$

and this is the condition for least fuel to intercept the target. Using a different formulation for the problem, Eggleston (ref. 1) has obtained a form of this equation. By comparison with equation (11), equation (13) represents the time to rendezvous with least fuel only in the case $E = 0$; that is, when the vehicles are already on a collision course. This result also shows that if the vehicles are on a collision course, there is no modification to the interception course that will result in a fuel saving for the complete rendezvous maneuver.

In general, equation (11) gives the optimum time or true rendezvous anomaly for least fuel when there are two impulses, with the first impulse applied following target acquisition. The magnitude of the first impulse, ΔV_1, for the optimum angle $w_o\tau$, is

$$\Delta V_1 = \frac{E}{w_oS + E}\sqrt{\dot{S}^2 + (w_oS + E)^2} \tag{14}$$

to which corresponds a terminal impulse of magnitude

$$\Delta V_o = \frac{w_o S}{w_o S + E} \sqrt{\dot{S}^2 + (w_o S + E)^2} \tag{15}$$

so that the total rendezvous impulse, ΔV, is

$$\Delta V = \sqrt{\dot{S}^2 + (w_o S + E)^2} \tag{16}$$

If, instead of the optimum fuel solution, the optimum energy solution (eq. (12)) is used, then the total impulse is

$$(\Delta V)_e = \Delta V_1 + \Delta V_o = \sqrt{\frac{\dot{S}^2}{4} + E^2} + \sqrt{\frac{\dot{S}^2}{4} + w_o^2 S^2} \tag{17}$$

so that, whenever $\dot{S}^2 \gg \max(E^2, w_o^2 S^2)$, comparison of equations (16) and (17) gives

$$(\Delta V)_e - \Delta V = \left(\dot{S} + \frac{E^2 + w_o^2 S^2}{\dot{S}}\right) - \left[\dot{S} + \frac{(w_o S + E)^2}{2\dot{S}}\right]$$

$$+ \text{ higher order terms}$$

or

$$(\Delta V)_e - \Delta V = \frac{(w_o S - E)^2}{2\dot{S}} + 0 \left[\frac{(w_o S - E)^4}{\dot{S}^3}\right] \tag{18}$$

This result shows that if E is larger than $w_o S$, but small compared to $\dot{S}$, it may be unnecessary to wait until the optimum time for least fuel, since for the least-energy rendezvous the additional fuel penalty is extremely small. For example, if $\dot{S} = 2000$ fps, $w_o S = 200$ fps, and $E = 400$ fps, and the target is in a near-earth satellite orbit, the cost of a rendezvous in the time for least fuel of 6 minutes is about 2090 fps, and in the least energy time of 4 minutes the cost is about 2100 fps. However, if the rendezvous is made in 2 minutes, which is the optimum time for intercept given by the solution of equation (13), the cost is up to 2410 fps, a significant increase. This increase follows since, if $(\Delta V)_I$ is the total impulse when the optimum intercept solution is used, the difference between the two increments is

$$(\Delta V)_I - \Delta V = E - \frac{E^2 + 2w_o SE}{2\dot{S}} + 0\left[\frac{(w_o S - E)^4}{\dot{S}^3}\right] \tag{19}$$

This equation shows that the two impulses differ by the first-order term E.

Coplanar specialization.- The equations describing the rendezvous maneuvers can be simplified still further if the specialization of coplanar orbits is considered. This specialization is of some interest. For example, many rendezvous maneuvers will be based upon compatible orbits (ref. 2) which tend to reduce the rendezvous to a coplanar maneuver. For the coplanar case, $\underline{\Omega} \times \underline{S_R}$ is parallel to $(\underline{V_{R_1}} \times \underline{S_R}) \times \underline{S_R}$ so that the expression for E^2 reduces to a perfect square:

$$E^2 = (V \cos \gamma - w_o \rho)^2$$

where $\rho^2 = x^2 + y^2$. Equation (11) now may be restated,

$$\left.\begin{aligned} \cot w_o \tau &= - \frac{V \sin \gamma}{2w_o \rho - V \cos \gamma} && \text{if} \quad w_o \rho > V \cos \gamma \\ w_o \tau &= \pi/2 + \gamma && \text{if} \quad w_o \rho \leq V \cos \gamma \end{aligned}\right\} \tag{20}$$

Usually, the latter case, $V \cos \gamma \geq w_o \rho$, will be of interest. Then an economical rendezvous is achieved simply by setting the true rendezvous anomaly equal to the angle between the relative range and the relative velocity. The initial and terminal impulses are of magnitudes,

$$\Delta V_1 = (V \cos \gamma - w_o \rho) \sec \gamma \; ; \quad \Delta V_o = w_o \rho \sec \gamma \tag{21}$$

respectively, and the total impulse, $\Delta V = \Delta V_o + \Delta V_1$, is just the relative velocity; that is,

$$\Delta V = V \tag{22}$$

Navigation considerations.- For navigation purposes, it is necessary to know both the magnitude and the direction of the velocity impulses required. The direction cosines of the impulses, l, m, n initially and l_o, m_o, n_o at interception, are

$$\begin{bmatrix} l \\ m \\ n \end{bmatrix} = -\frac{w_o}{\Delta V_1}\begin{bmatrix} \cot w_o\tau + \frac{\dot{x}_1}{w_o x} & -1 & 0 \\ 1 & \cot w_o\tau + \frac{\dot{y}_1}{w_o y} & 0 \\ 0 & 0 & \cot w_o\tau + \frac{\dot{z}_1}{w_o z} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (23)$$

and

$$\begin{bmatrix} l_o \\ m_o \\ n_o \end{bmatrix} = \frac{w_o}{\Delta V_o}\begin{bmatrix} \cot w_o\tau & 1 & 0 \\ -1 & \cot w_o\tau & 0 \\ 0 & 0 & \sqrt{1 + \cot^2 w_o\tau} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (24)$$

In the coplanar problem with $w_o\rho \leq V\cos\gamma$, the least-fuel solution corresponds to $\cot w_o\tau = -\tan\gamma$ so that $\sqrt{1 + \cot^2 w_o\tau} = \sec\gamma$ and the transforming matrices above are especially simple.

It is not the intention here to deliberate upon the manner by which the target intelligence is obtained or upon the transformations that are needed to orient the problem in terms of the interceptor's coordinate reference system. However, it would appear that the quantities needed to evaluate and determine the rendezvous maneuvers subsequent to target acquisition can be obtained readily. Thus, the vector $\underline{\Omega}$ would be predetermined based upon an accurate target ephemeris, and stored in the interceptor's inertial guidance system. The magnitude of the relative range, S, and the range rate, $\dot{S} = V\sin\gamma$, could be obtained by an on-board precision Doppler radar, and the direction fixed by optical means. There remain the components of relative velocity perpendicular to the relative range, $\underline{S}_R$. These could be obtained by measuring the rate of change of direction of the relative range vector immediately following target acquisition.

Noncircular Target Orbits

The foregoing analysis is restricted to circular target orbits. In the following section, this restriction will be dropped. In addition, it was assumed early in the previous analysis that gravity was constant over the altitude range of interest (in particular, that $w_o y \ll \dot{x}$). Although this assumption is necessary to this analysis, it is not essential to a closed solution to the motion for a circular target orbit (e.g., ref. 1). This assumption will not be made in the more comprehensive solution now to be developed. This solution is reduced to the circular case in appendix B.

Although the target orbits now considered are noncircular, some simplifications of the orbit equations are possible. For example, it has been shown in reference 3 that a necessary condition for rendezvous with a minimum expenditure of fuel, when the interceptor is launched from the surface of the planet, is that the rendezvous should occur at an apsis of the target's elliptical orbit. If the target orbit is an open conic, it can be shown that rendezvous should take place at the target periapsis and as near the periapsis of the interceptor's trajectory as optimum burnout altitude considerations will allow. Therefore, if the target orbit is a conic, we shall set the constraint upon the motion that the rendezvous maneuvers be in the vicinity of an apsis. We will consider the true anomaly, v, is small such that $\cos v = 1$ and $\sin v = v$. These approximations are valid near periapsis and we will make them valid near apoapsis of an elliptical target orbit by measuring v from this point and taking the eccentricity e as negative under this condition. The equations of motion are then:

$$\left.\begin{aligned}
\ddot{x} - 2w_o(1 + e)^2\dot{y} + 2w_o^2(1 + e)^3 evy - w_o^2 e(1 + e)^3 x &= 0 \\
\ddot{y} + 2w_o(1 + e)^2\dot{x} - 2w_o^2(1 + e)^3 evx - w_o^2(3 + e)(1 + e)^3 y &= 0 \\
\ddot{z} + w_o^2(1 + e)^3 z &= 0
\end{aligned}\right\} \qquad (25)$$

where

$$v = w_o(1 + e)^2(t - t_o) + O(t - t_o)^2$$

with t_o the time to the apsis in the target orbit. Terms of order $w_o^2x^2$ have been neglected. If now terms of order w_o^2vx may be neglected also, the equations integrate immediately to:

$$q = e^{w_1\tau}\left(A_q\, e^{w_2\tau} + B_q\, e^{-w_2\tau}\right) + e^{-w_1\tau}\left(C_q\, e^{w_2\tau} + D_q\, e^{-w_2\tau}\right) \qquad (26)$$

where

$$q = x \text{ or } y$$

$$z = -\frac{\dot{z}_o \sin w_z\tau}{w_z} \qquad (27a)$$

$$w_1^2 = w_z^2\left[\frac{\sqrt{e(3+e)}}{2} - \frac{1+2e}{4}\right] \qquad w_2^2 = -w_z^2\left[\frac{\sqrt{e(3+e)}}{2} + \frac{1+2e}{4}\right] \qquad (27b)$$

$$w_z^2 = w_o^2\,(1+e)^3 \qquad (27c)$$

$$2A_q = -(\lambda_q - \rho_q)\,\dot{q}_o + \alpha\dot{Q}_o \qquad 2C_q = (\lambda_q + \rho_q)\,\dot{q}_o - \alpha\dot{Q}_o \qquad (27d)$$

$$2B_q = -(\lambda_q + \rho_q)\,\dot{q}_o - \alpha\dot{Q}_o \qquad 2D_q = (\lambda_q - \rho_q)\,\dot{q}_o + \alpha\dot{Q}_o \qquad (27e)$$

$$Q_o = \text{complement of } q_o = \begin{cases} +y & \text{if } q \equiv x \\ -x & \text{if } q \equiv y \end{cases} \qquad (27f)$$

$$\alpha = \frac{w_z\sqrt{1+e}}{2w_1w_2} \qquad (27g)$$

$$\lambda_q = \frac{1+\sqrt{\beta_q}}{4w_1} \qquad \rho_q = \frac{-1+\sqrt{\beta_q}}{4w_2} \qquad (27h)$$

$$\beta_x = \frac{3+e}{e} \qquad \beta_y = \frac{e}{3+e} \qquad (27i)$$

The computation has been carried out in negative time $\tau = -t$ with the boundary conditions $\underline{S}_R = 0, 0, 0$, $\underline{V}_{R_o} = \dot{x}_o, \dot{y}_o, \dot{z}_o$, applied at $t = 0$.

The vector $\underline{V}_R(\dot{x}, \dot{y}, \dot{z})$ is the required initial velocity at the position $\underline{S}_R(x, y, z)$ in order to intercept the target in time τ, and the vector $\underline{V}_{Ro}(\dot{x}_o, \dot{y}_o, \dot{z}_o)$ is the terminal impulse needed to bring the vehicles to relative rest at the intercept position, that is, to complete the rendezvous. The velocity components are related to the space coordinates by

$$\begin{bmatrix} \dot{x}_o \\ \dot{y}_o \\ \dot{z}_o \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; ; \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \left(\frac{dP}{dt}\right) P^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{28}$$

where the matrix P is defined by equations (26) and (27). We find that

$$P^{-1}\Delta = - \begin{bmatrix} a_{xx} & -a_{xy} & 0 \\ a_{xy} & a_{yy} & 0 \\ 0 & 0 & -w_z\Delta \operatorname{cosec} w_z\tau \end{bmatrix}$$

where

$$a_{xx} = (\lambda_y + \rho_y) \sinh (w_1 - w_2) \tau + (\lambda_y - \rho_y) \sinh (w_1 + w_2)\tau$$

$$a_{xy} = \alpha \left[\cosh (w_1 - w_2)\tau - \cosh (w_1 + w_2)\tau\right]$$

$$a_{yy} = (\lambda_x + \rho_x) \sinh (w_1 - w_2)\tau + (\lambda_x - \rho_x) \sinh (w_1 + w_2)\tau$$

$$\Delta = \frac{2 + \beta_x + \beta_y}{4w_1^2} \sinh^2 w_1\tau + \frac{2 - \beta_x - \beta_y}{4w_2^2} \sinh^2 w_2\tau$$

$$\frac{dP}{dt} = \begin{bmatrix} b_{xx} & b_{xy} & 0 \\ -b_{xy} & b_{yy} & 0 \\ 0 & 0 & -\cos w_z\tau \end{bmatrix}$$

where

$$b_{qq} = (\lambda_q + \rho_q)(w_1 - w_2) \cosh (w_1 - w_2)\tau$$
$$+ (\lambda_q - \rho_q)(w_1 + w_2) \cosh (w_1 + w_2)\tau$$

$$q = x \text{ or } y$$

$$b_{xy} = \alpha \left[(w_1 - w_2) \sinh (w_1 - w_2)\tau - (w_1 + w_2) \sinh (w_1 + w_2)\tau\right]$$

Since we are concerned with small motion near an apsis, the intractable expressions for the magnitudes of the impulses that result from equation (28) may be replaced by the initial terms of their expansions in terms of $w_z\tau$. After some manipulation, we obtain:

$$\left.\begin{aligned} (\Delta V_o)^2 &= (w_z S \operatorname{cosec} w_z\tau)^2 - w_z^2 y^2 + 0\ (w_z\tau) \\ (\Delta V_1)^2 &= (w_z S \cot w_z\tau + \dot{S})^2 + F^2 + 0\ (w_z\tau) \end{aligned}\right\} \quad (29)$$

where it can be shown that

$$F^2 = E^2 + w_z^2 y^2 \geq 0$$

Clearly, the velocity impulse to intercept the target, ΔV, is minimized by choice of

$$\cot w_z\tau = - \frac{\dot{S}}{w_z S} \quad (30)$$

which is identical to equation (13) with w_z replacing w_o.

Previously, we called the quantity,

$$\frac{1}{2}(\Delta V_e)^2 = \frac{1}{2}\left[(\Delta V_o)^2 + (\Delta V_1)^2\right]$$

the equivalent energy of the transfer impulses. It has a minimum when

$$\cot w_z\tau = -\frac{\dot{S}}{2w_zS} \tag{31}$$

which is identical to equation (12) with w_z replacing w_o.

Next, let us form the total impulse, $\Delta V = \Delta V_o + \Delta V_1$, which is a measure of the fuel consumed in the rendezvous:

$$\Delta V = \sqrt{(w_zS \operatorname{cosec} w_z\tau)^2 - w_z^2y^2} + \sqrt{(w_zS \cot w_z\tau + \dot{S})^2 + F^2}$$

Differentiation with respect to $w_z\tau$ with the dividend set to zero gives

$$\cot w_z\tau = -\frac{\dot{S}}{w_zS + D} \tag{32}$$

where

$$D^2 = \left(\frac{S^2}{S^2 - y^2}\right) F^2$$

as the condition for least fuel to rendezvous. This equation is similar to equation (11) for the circular case, with E replaced by,

$$D = E\left[\frac{1 + \left(\frac{w_zy}{E}\right)^2}{1 - \left(\frac{y}{S}\right)^2}\right]^{1/2}$$

Under the least-fuel condition (eq. (32)), the total impulse, ΔV, has a value given by

$$\Delta V = \sqrt{\dot{S}^2 + \left(\sqrt{w_z^2 S^2 - w_z^2 y^2} + F\right)^2} \tag{33}$$

Equations (28), (29), (32), and (33) provide the information needed to navigate a least-fuel rendezvous based upon two impulses, with the first impulse applied following target acquisition, and the target orbit either circular or near an apsis of any conic orbit.

RESULTS AND DISCUSSION

With the mathematics of the rendezvous problem developed, it is now appropriate to discuss some of the implications of the results obtained. Many features of the rendezvous problem can be illustrated with the simple case of a circular target orbit and a coplanar interceptor. An example of this type has been examined for the conditions of a target in a 1000-mile orbit, a separation at time of first impulse of 40 miles, and a relative velocity at this time of 1350 feet per second. The various velocity impulses associated with rendezvous for these conditions are shown as a function of time to rendezvous in figure 2. The initial impulse, ΔV_1, the terminal impulse, ΔV_o, their total, ΔV, and the equivalent energy impulse ΔV_e are all presented, and the minimum values are indicated with arrows on the curves. The initial impulse, ΔV_1, which is also that required to intercept, has a very pronounced minimum. This result is a general characteristic; no matter what the initial conditions of the relative motion, there is a distinct optimum time to intercept. If we deviate too much on either side of that optimum, we will pay a considerable penalty in total velocity and, hence, in total fuel requirement. The terminal velocity curve, ΔV_o, is approximately a simple rectangular hyperbola, so that the total velocity impulse requirement, ΔV, has a minimum displaced from that of ΔV_1 to a later time. In this case, the minimum is 1350 feet per second, equal to the initial relative velocity. The minimum is fairly shallow in typical situations, and if it occurs at a time very much in excess of the time for the minimization of the equivalent energy of the maneuver, ΔV_e, it is generally unnecessary to wait for the fuel minimum. A rendezvous maneuver that takes a time equal to the minimization time for ΔV_e will in these cases consume very little fuel in excess of the true minimum fuel requirement. Hence, an economical rendezvous can often be achieved in a time that is not less than the time for least-fuel interception, and not more than the time for least-equivalent energy rendezvous, about twice the least intercept time.

The results presented in figure 2 illustrate the cost of a rendezvous maneuver for fixed initial conditions and thus for a fixed time for initiation of the maneuver. It is apparent that in the interests of economy there will be occasions when it will be advantageous to wait before starting the maneuver. Consider, for example, a typical rendezvous situation. The interceptor is launched to burnout at the periapsis of a transfer ellipse and then, in the ideal case, coasts toward an apoapsis cotangential with the target in its orbit. Suppose again, that the target orbit is circular and that the ideal interceptor orbit is coplanar with it. It can be proved that the magnitude of the relative velocity vector, V, is a minimum at the cotangential transfer point. Now let us refer to equation (22), which is valid for this special case, with the assumption that $w_0 y \ll \dot{x}$. The cost of the rendezvous maneuver is, by this equation, the relative velocity. Thus, when we are off course, the rendezvous initial impulse is applied at the best possible position in the uncorrected relative motion if it is called for when the relative velocity, V, is a minimum. In this simple case, provided we wish to rendezvous with the target rather than merely intercept it, the relative velocity is a more useful quantity than the error velocity, E, since its temporal behavior tells us when to initiate a rendezvous maneuver and its magnitude tells us the total cost. For an interception, this information is contained in the error velocity E and its rate of change.

Obviously, we can generalize these results. For the general problem that we have solved, we need to compute ΔV from equation (33) and measure its rate of change $d\Delta V/dt$. If ΔV is increasing at target acquisition, then we apply immediately the initial impulse, ΔV_1, for a locally least-fuel rendezvous, given by equations (29) and (32) combined, and effect a two-impulse rendezvous in the way described. If ΔV is decreasing, then the situation is continually improving and the cost of the rendezvous is continually diminishing, so we take no action. We continue to take no action until the quantity $d\Delta V/dt$ becomes zero, in other words, until ΔV becomes stationary. Then, before ΔV starts to increase, we apply the optimum ΔV_1. For the interception case, the principle for economical interception may be stated similarly to that for rendezvous, with ΔV and $d\Delta V/dt$ replaced by the error velocity, F, and its rate of change dF/dt.

The method for economical rendezvous is a multi-impulse scheme since, if the first ΔV_1 is delivered in error, the guidance logic based upon the continuous computation of ΔV and $d\Delta V/dt$ will call for an additional impulse as soon as an error appears. To anticipate an off-course error, and to attempt to correct it without using the guidance logic, may result in a substantial increase in the rendezvous fuel requirement. It is appropriate therefore to call this guidance logic the principle of least action for economical rendezvous.

To illustrate some of the advantages of this principle, the results contained in figure 3 have been prepared. In figure 3(a), a circular target orbit of 1,000-mile altitude and a coplanar interceptor are considered. The relative range, relative velocity (which in this case is also the instantaneous least-fuel velocity requirement for rendezvous), and the error velocity are shown as functions of the time at which the maneuver is initiated. For the time period considered, the rendezvous least-fuel requirement is shown to vary from about 3,000 feet per second to a minimum of about 950 feet per second, depending upon the time at which we apply the initial rendezvous impulse. Observe that the minimum in the ΔV curve occurs at a different time than that of the error velocity. For the example shown in figure 3(a), the rendezvous was considered to be initiated at the time when ΔV was a minimum. After the initial rendezvous has been applied ΔV becomes constant if the initial impulse is delivered without error. If an error has occurred it can be distinguished by the fact that ΔV is not constant following the impulse. With the guidance logic discussed earlier, therefore, we have a multi-impulse scheme for economical rendezvous.

In figure 3(b), a similar set of results[2] is shown for the more general case of a noncircular orbit and a noncoplanar interceptor. In this case the difference between the relative velocity and the least-fuel velocity requirement is displayed. It is noted, however, that for situations near the point where the least-fuel requirement is a minimum there is little difference between the two velocities. This result indicates that the simplified analysis may be particularly useful for the guidance in the final phases of a rendezvous maneuver when the computation time required in the solution of the guidance equations may be an important consideration.

It should be noted that the results presented in figure 3 represent only example situations. For this reason, obviously, the initial assumptions affect the magnitudes of the quantities involved. However, the analytic characteristics are independent of these assumptions.

CONCLUDING REMARKS

Some of the problems associated with the rendezvous of two space vehicles have been considered. In this study, the equations of relative motion for rendezvous with impulsive velocity changes were developed and solved for the case of a circular target orbit and for the case of motion near the apsis of any conic target orbit. The maneuvers associated with least-fuel interception, least-fuel rendezvous, and least-energy rendezvous were determined from the solutions. The impulse requirement for least-fuel interception was found to be relatively distinct; however, that associated with least-fuel rendezvous was not. For this reason it

[2]The rendezvous maneuvers are omitted for clarity but are similar to figure 3(a).

was indicated that the time required for a rendezvous maneuver often may be shortened significantly from that for least fuel without incurring a significant fuel penalty. It was further shown that an economical rendezvous depends not only upon the time taken for the rendezvous maneuver from a given set of initial conditions; but in addition, it depends on the selection of the initiation point in the uncorrected relative motion. From these considerations, a guidance logic for economical rendezvous was developed. This sytem of logic is a multi-impulse scheme depending upon a priori knowledge of the target orbit parameters and on the measurement of the relative range and relative velocity vectors following target acquisition. With the aid of the results of the analysis several sample rendezvous maneuvers were examined. It was found that the penalty for commencing a rendezvous maneuver at a position significantly different from the optimum position may be substantial.

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APPENDIX A

PROPAGATION OF ERRORS AT LAUNCH INTO MISS DISTANCE AT THE TARGET

Consider a typical rendezvous situation in which the interceptor is launched to burnout at the perigee of a transfer ellipse, and coasts, in the ideal case, to an apogee cotangential with the target in its orbit. Suppose an error is made in the initial burnout conditions. How does this influence the miss distance at the target? The relevant partial derivatives of the transfer orbit are:

$$\frac{\partial r_o}{\partial V_b} = \frac{(r_o + r_b)^2 V_b}{\mu} = \frac{4}{\pi}\tau_o\sqrt{\frac{r_o}{r_b}} \tag{A1}$$

$$\frac{\partial r_o}{\partial r_b} = \left(\frac{r_o}{r_b}\right)^2 + 2\left(\frac{r_o}{r_b}\right) \tag{A2}$$

$$\frac{\partial \tau_o}{\partial r_b} = \frac{3\pi}{4}\sqrt{\frac{r_o + r_b}{2\mu}} \tag{A3}$$

$$\frac{\partial \tau_o}{\partial V_b} = 3\tau_o\sqrt{\frac{r_o}{r_b}}\sqrt{\frac{r_o + r_b}{2\mu}} \tag{A4}$$

where r_b is the radius and V_b the velocity of the interceptor at booster cutoff; r_o is the radius of the interceptor at the target and τ_o is the time taken for the coast along the Hohmann transfer ellipse from r_b to r_o.

If, at burnout, the 3σ uncertainties in velocity, altitude, and time are 20 fps, 5 miles, and 2 seconds, and the nominal burnout altitude is 70 miles, the following errors at the target are obtained for various target altitudes, h:

h,miles	s,miles	y,miles
135	39	20
200	40	20
300	41	21
400	41	22
1000	47	26

where S is the 3σ value for the miss distance at the target, and y is the 3σ value for the uncertainty in altitude relative to the target.

APPENDIX B

GENERAL FIRST-ORDER SOLUTION TO THE CIRCULAR TARGET ORBIT PROBLEM

In the case of a circular target orbit, the relative vector equation of motion reduces to:

$$\frac{d^2 \underline{S}_R}{dt^2} + 2(\underline{\Omega} \times \underline{V}_R) + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = \triangle \left(\frac{\mu}{r}\right) + \frac{\underline{T}}{m} \tag{B1}$$

where $\underline{S}_R$ has the components x, y, z; $\underline{V}_R$ the components $\dot{x}, \dot{y}, \dot{z}$; and $\underline{\Omega}$ the components $0, 0, w$. Considering impulsive motion only, we may set $\underline{T} \equiv 0$.

The scalar equations of motion are now:

$$\left.\begin{aligned} \ddot{x} - 2w\dot{y} - w^2 x &= -\frac{\mu}{r^3} x \\ \ddot{y} + 2w\dot{x} - w^2 (y + r_o) &= -\frac{\mu}{r^3} (y + r_o) \\ \ddot{z} &= -\frac{\mu}{r^3} z \end{aligned}\right\} \tag{B2}$$

where

$$r^2 = r_o^2 \left(1 + 2 \frac{y}{r_o} + \frac{x^2 + y^2 + z^2}{r_o^2}\right) \tag{B3}$$

and

$$w^2 r_o^3 = \mu \tag{B4}$$

If

$$\frac{S_R^2}{r_o^2} = \frac{x^2 + y^2 + z^2}{r_o^2}$$

is small,

$$r^{-3} \simeq r_o^{-3}\left(1 - 3\,\frac{y}{r_o}\right)$$

so that

$$-\frac{\mu}{r^3}\,x \simeq -w^2\,x\ , \qquad -\frac{\mu}{r}\,z \simeq w^2 z\ ,$$

and

$$-\frac{\mu}{r^3}\,(y + r_o) = -w^2(y + r_o)\left(1 - 3\,\frac{y}{r_o}\right) \simeq -w^2\,(y + r_o - 3y)$$

whence, to the first order of approximation,

$$\left.\begin{aligned} \ddot{x} &= 2w\dot{y} \\ \ddot{y} &= -2w\dot{x} + 3w^2 y \\ \ddot{z} &= -w^2 z \end{aligned}\right\} \qquad \text{(B5)}$$

Applying the boundary conditions at $t = 0$, $x = x_o$, $y = y_o$, $z = z_o$, $\dot{x} = \dot{x}_o$, $\dot{y} = \dot{y}_o$, and $\dot{z} = \dot{z}_o$, we obtain the solutions,

$$\left.\begin{aligned} x &= \left(x_o + 2\,\frac{\dot{y}_o}{w}\right) + (6wy_o - 3\dot{x}_o)t - 2\,\frac{\dot{y}_o}{w}\cos wt - \left(6y_o - 4\,\frac{\dot{x}_o}{w}\right)\sin wt \\ y &= \left(4y_o - 2\,\frac{\dot{x}_o}{w}\right) - \left(3y_o - 2\,\frac{\dot{x}_o}{w}\right)\cos wt + \frac{\dot{y}_o}{w}\sin wt \\ z &= z_o\cos wt + \frac{\dot{z}_o}{w}\sin wt \end{aligned}\right\} \qquad \text{(B6)}$$

which are valid for small departures ($S_R = o(r_o)$) from the target location.

If we compute in negative time we are able to set $x_o = 0$, $y_o = 0$, and $z_o = 0$ to obtain:

$$\left.\begin{aligned}
\frac{\dot{x}_o}{w} &= \frac{x \sin wt - 2y(1 - \cos wt)}{8(1 - \cos wt) - 3wt \sin wt} \\
\frac{\dot{y}_o}{w} &= \frac{2x(1 - \cos wt) + y(4 \sin wt - 3wt)}{8(1 - \cos wt) - 3wt \sin wt} \\
\frac{\dot{z}_o}{w} &= \frac{z}{\sin wt}
\end{aligned}\right\} \quad (B7)$$

$$\left.\begin{aligned}
\frac{\dot{x}}{w} &= \frac{x \sin wt + 2y\left[7(1 - \cos wt) - 3wt \sin wt\right]}{8(1 - \cos wt) - 3wt \sin wt} \\
\frac{\dot{y}}{w} &= \frac{-2x(1 - \cos wt) + y(4 \sin wt - 3wt \cos wt)}{8(1 - \cos wt) - 3wt \sin wt} \\
\frac{\dot{z}}{w} &= z\,\frac{\cos wt}{\sin wt}
\end{aligned}\right\} \quad (B8)$$

Hence, if initially the relative velocity vector at x_1, y_1, z_1 is $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$, we are able to compute the sum of the magnitudes, ΔV, of the initial impulse $\underline{\Delta V_1} = \dot{x} - \dot{x}_1, \dot{y} - \dot{y}_1, \dot{z} - \dot{z}_1$, and the terminal impulse $\underline{\Delta V_o} = \dot{x}_o, \dot{y}_o, \dot{z}_o$ that reduces the relative velocity vector at the position 0, 0, 0 to zero, in terms of the time to rendezvous, $\tau = -t$:

$$\Delta V = \Delta V_1 + \Delta V_o \equiv \Delta V\ (\tau)$$

The resultant expression for ΔV is cumbersome and difficult to handle analytically. However, we may approximate in the manner of equations (29), with the quantity w replacing w_z, in order to determine the position at which to rendezvous and the time to take. The actual impulses would then be determined by the exact equations (28), alternatively expressed for circular orbits in the equations above, with $t = -\tau$.

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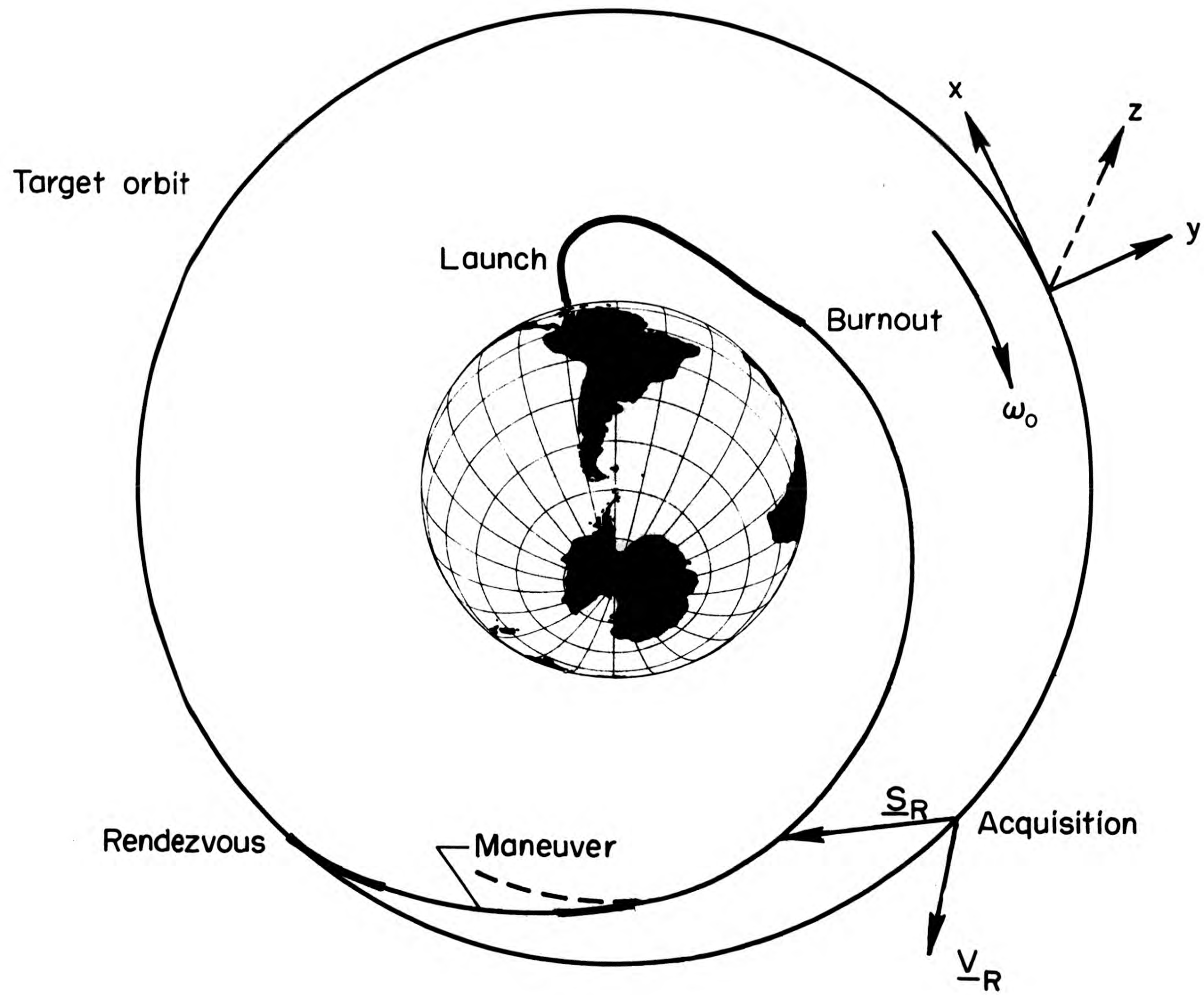
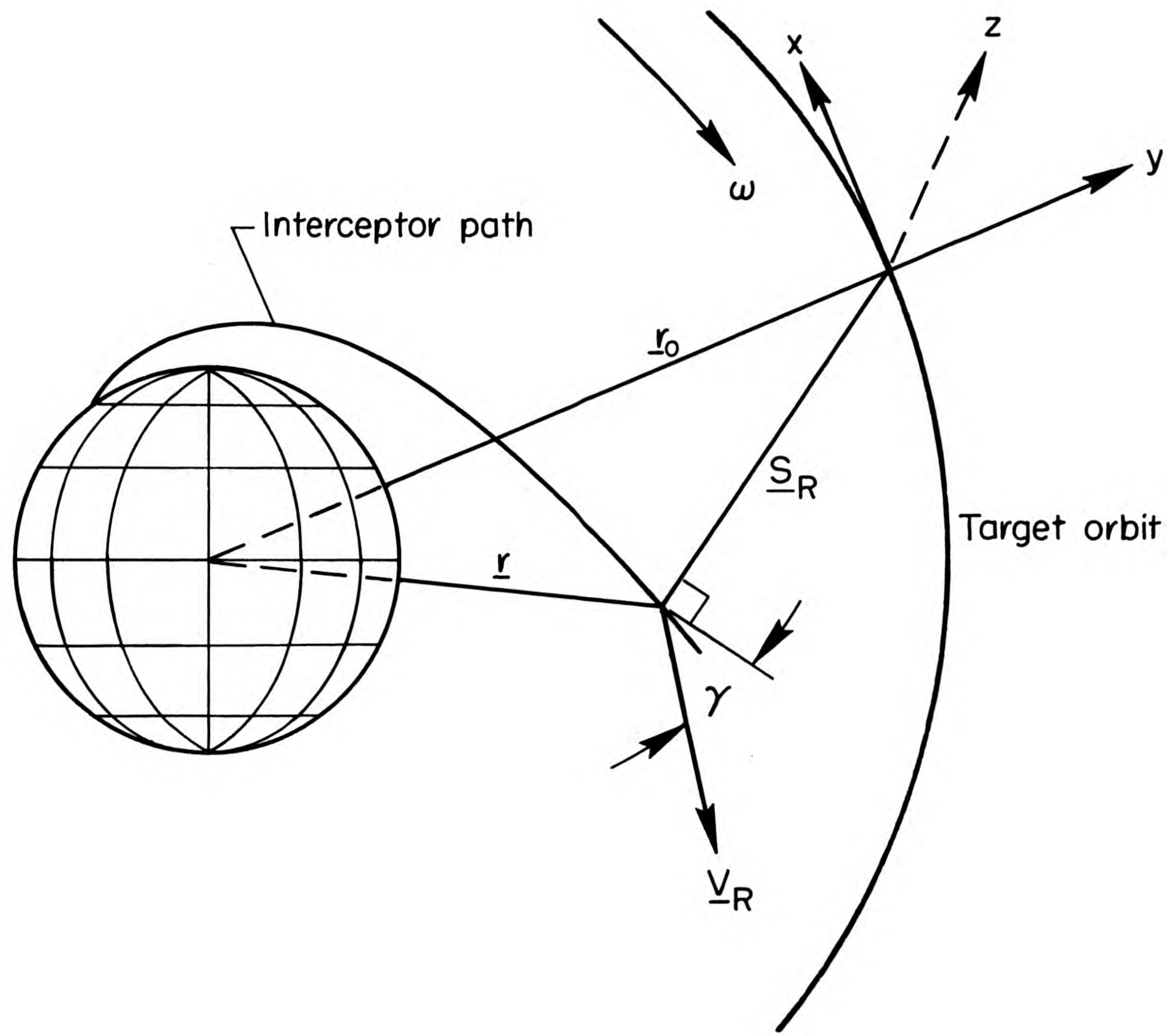


(a) Schematic drawing.

Figure 1.- A typical rendezvous maneuver.



(b) Notation.

Figure 1.- Concluded.

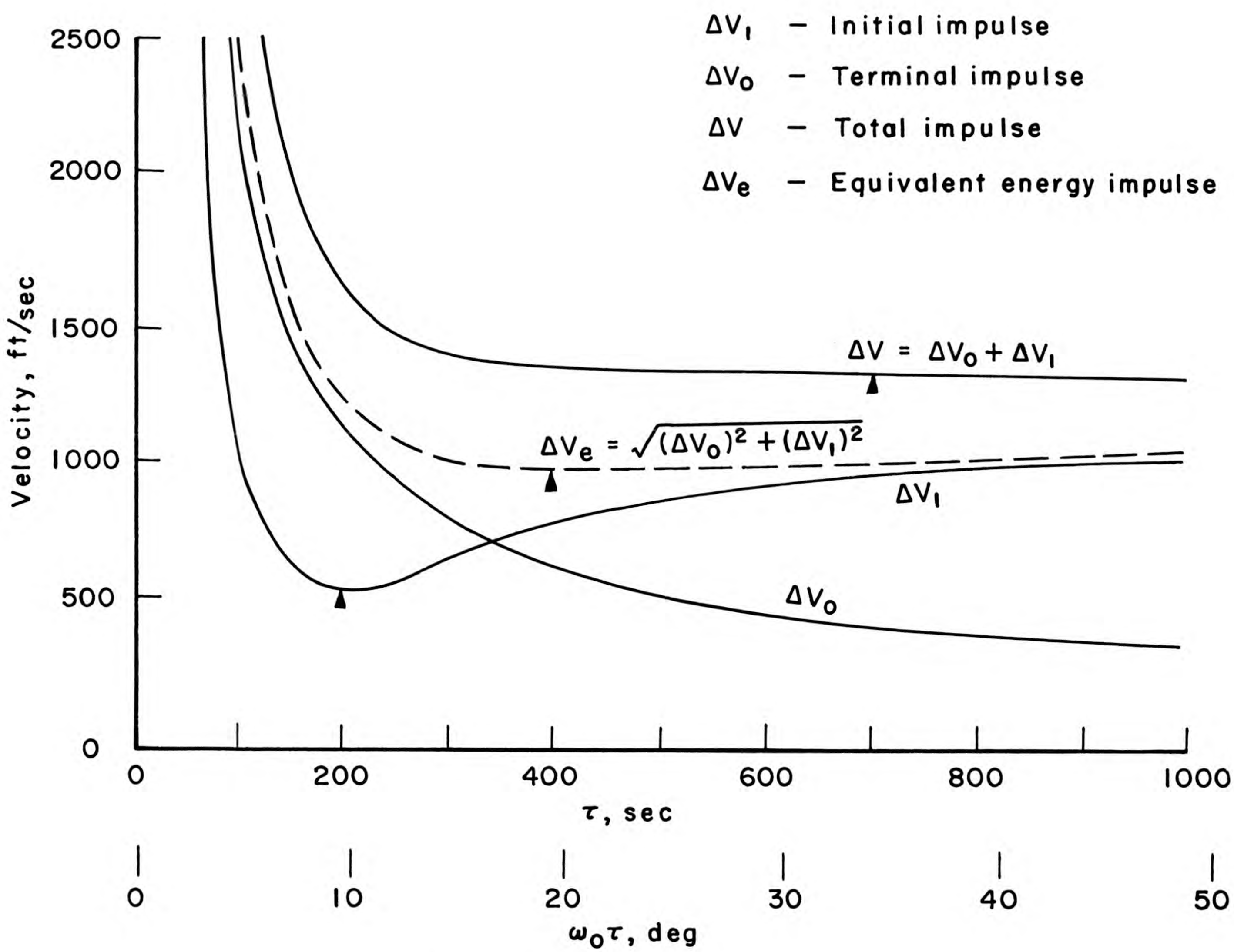
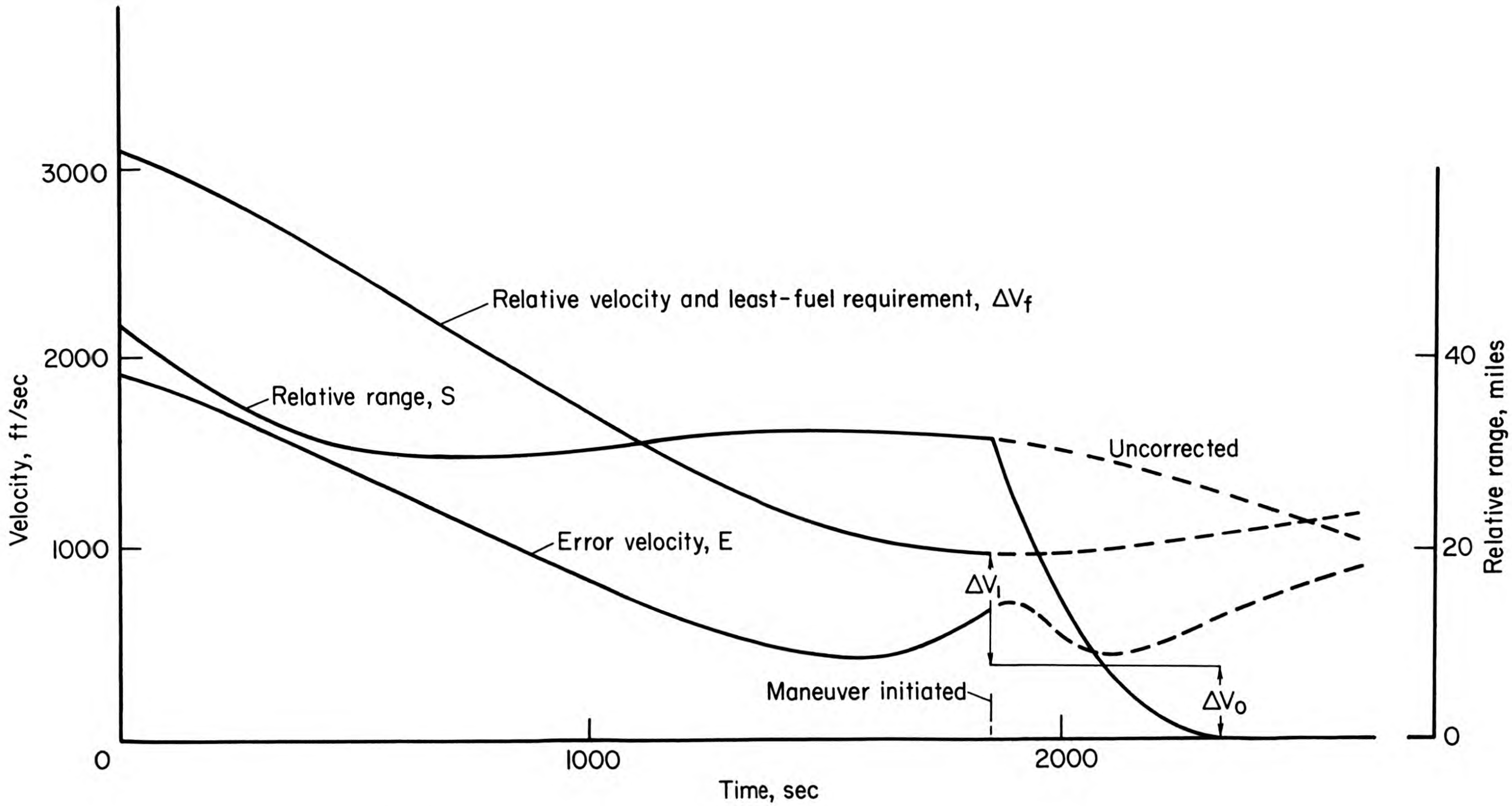
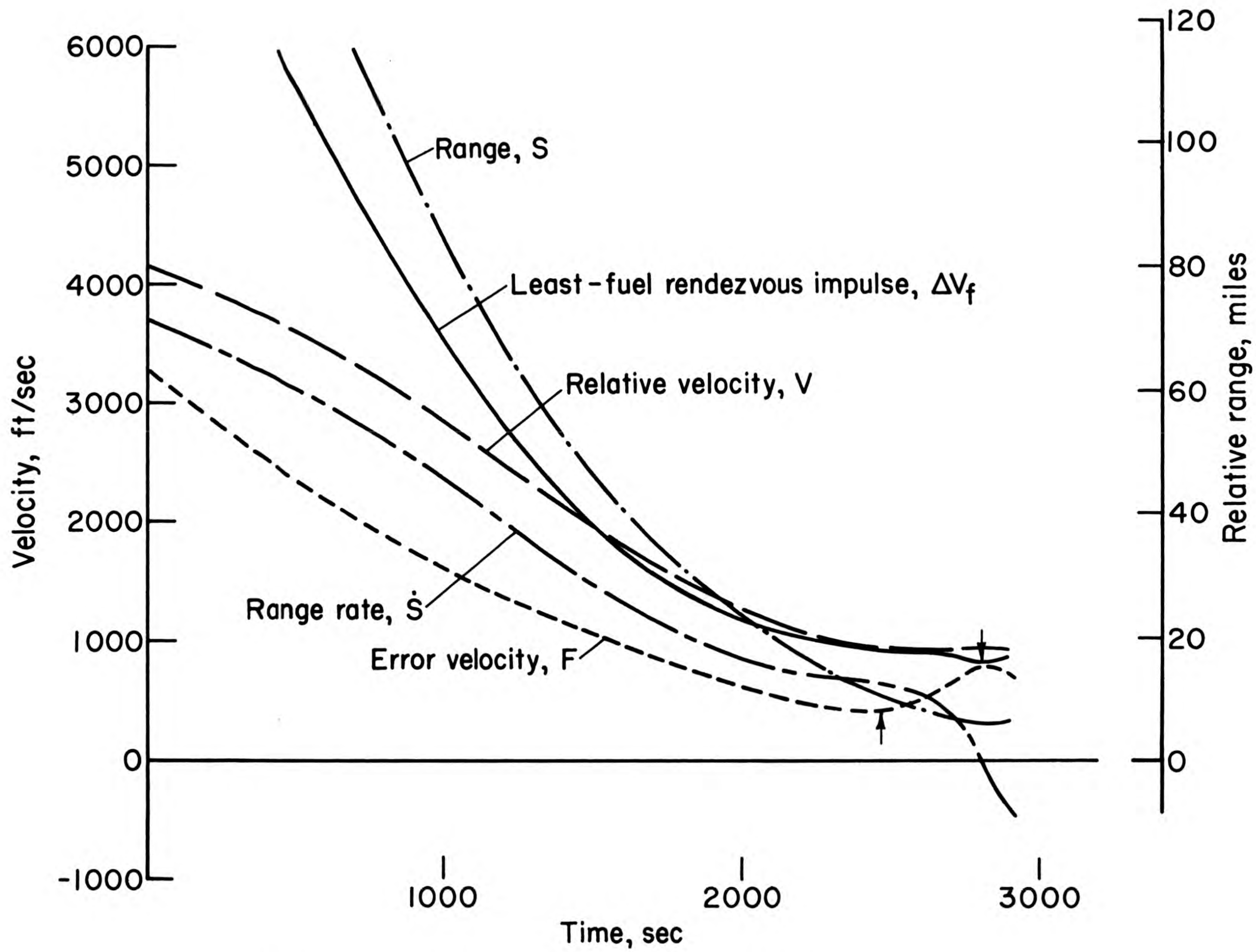


Figure 2.- Rendezvous and interception velocity impulses for 1000-mile circular target orbit and coplanar interception with initial range of 40 miles.



(a) Circular target orbit and coplanar interceptor.

Figure 3.- Principle of least action for initiating a rendezvous maneuver.



(b) Noncircular target orbit and noncoplanar interceptor.

Figure 3.- Concluded.

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